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Using Interval Recurrence Formula to Solve Two Improper Fractional Integrals

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Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we use interval recurrence formula to solve two improper fractional integrals. In fact, our result is a generalization of classical calculus result.**

Keywords: **Jumarie type of R-L fractional calculus, new multiplication, fractional analytic functions, interval recurrence formula, improper fractional integrals.**

I. INTRODUCTION

Fractional calculus is a natural extension of the traditional calculus. In fact, since the beginning of the theory of differential and integral calculus, several mathematicians have studied their ideas on the calculation of non-integer order derivatives and integrals. However, the application of fractional derivatives and integrals has been scarce until recently. In the last decade, fractional calculus are widely used in physics, mechanics, biology, electrical engineering, viscoelasticity, control theory, economics, and other fields [1-12].

However, fractional calculus is different from ordinary calculus. The definition of fractional derivative is not unique. Common definitions include Riemann Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative and Jumarie's modification of R-L fractional derivative [13-17]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on the Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, we use interval recurrence formula to solve the following two improper α-fractional integrals:

$$\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_\alpha\left(sin_\alpha(\theta^\alpha)\right)\right],$$

and

$$\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_\alpha\left(cos_\alpha(\theta^\alpha)\right)\right].$$

Where $0 < \alpha \leq 1$. Moreover, our result is a generalization of ordinary calculus result.

II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper.

Definition 2.1 ([18])**:** Let $0 < \alpha \leq 1$, and θ_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{\theta_0}D_\theta^\alpha\right)[f(\theta)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\theta}\int_{\theta_0}^{\theta}\frac{f(t)-f(\theta_0)}{(\theta-t)^\alpha}dt\ , \tag{1}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{\theta_0}I_\theta^\alpha\right)[f(\theta)] = \frac{1}{\Gamma(\alpha)}\int_{\theta_0}^{\theta}\frac{f(t)}{(\theta-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

In the following, some properties of Jumarie type of R-L fractional derivative are introduced.

Proposition 2.2 ([19])**:** *If* α,β,θ_0,c *are real numbers and* $\beta\geq\alpha>0$, *then*

$$\left({}_{\theta_0}D_\theta^\alpha\right)\left[(\theta-\theta_0)^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(\theta-\theta_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{\theta_0}D_\theta^\alpha\right)[c] = 0. \tag{4}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([20])**:** If θ,θ_0, and a_n are real numbers for all n, $\theta_0\in(a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(\theta^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(\theta-\theta_0)^{n\alpha}$ on some open interval containing θ_0, then we say that $f_\alpha(\theta^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([21])**:** Let $0<\alpha\leq 1$, and θ_0 be a real number. If $f_\alpha(\theta^\alpha)$ and $g_\alpha(\theta^\alpha)$ are two α-fractional analytic functions defined on an interval containing θ_0 ,

$$f_\alpha(\theta^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(\theta-\theta_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(\theta^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(\theta-\theta_0)^{n\alpha}\ . \tag{6}$$

Then we define

$$\begin{aligned} & f_\alpha(\theta^\alpha)\otimes_\alpha g_\alpha(\theta^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(\theta-\theta_0)^{n\alpha}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(\theta-\theta_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(\theta-\theta_0)^{n\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(\theta^\alpha)\otimes_\alpha g_\alpha(\theta^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha\right)^{\otimes_\alpha n}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha\right)^{\otimes_\alpha n}. \end{aligned} \tag{8}$$

Definition 2.5 ([22])**:** If $0<\alpha\leq 1$, and $f_\alpha(\theta^\alpha)$, $g_\alpha(\theta^\alpha)$ are two α-fractional analytic functions defined on an interval containing θ_0 ,

$$f_\alpha(\theta^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(\theta-\theta_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha\right)^{\otimes_\alpha n}, \tag{9}$$

$$g_\alpha(\theta^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(\theta-\theta_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(\theta-\theta_0)^\alpha\right)^{\otimes_\alpha n}. \tag{10}$$

The compositions of $f_\alpha(\theta^\alpha)$ and $g_\alpha(\theta^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(\theta^\alpha) = f_\alpha\big(g_\alpha(\theta^\alpha)\big) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \big(g_\alpha(\theta^\alpha)\big)^{\otimes_\alpha n}, \tag{11}$$

and

$$(g_\alpha \circ f_\alpha)(\theta^\alpha) = g_\alpha\big(f_\alpha(\theta^\alpha)\big) = \sum_{n=0}^{\infty} \frac{b_n}{n!} \big(f_\alpha(\theta^\alpha)\big)^{\otimes_\alpha n}. \tag{12}$$

Definition 2.6 ([23])**:** Let $0 < \alpha \le 1$. If $f_\alpha(\theta^\alpha), g_\alpha(\theta^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha \circ g_\alpha)(\theta^\alpha) = (g_\alpha \circ f_\alpha)(\theta^\alpha) = \frac{1}{\Gamma(\alpha+1)}\theta^\alpha. \tag{13}$$

Then $f_\alpha(\theta^\alpha), g_\alpha(\theta^\alpha)$ are called inverse functions of each other.

Definition 2.7 ([24])**:** If $0 < \alpha \le 1$, and θ is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(\theta^\alpha) = \sum_{n=0}^{\infty} \frac{\theta^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes_\alpha n}. \tag{14}$$

And the α-fractional logarithmic function $Ln_\alpha(\theta^\alpha)$ is the inverse function of $E_\alpha(\theta^\alpha)$. On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(\theta^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes_\alpha 2n}, \tag{15}$$

and

$$sin_\alpha(\theta^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{16}$$

Definition 2.8 ([25])**:** Let $0 < \alpha \le 1$, and $f_\alpha(\theta^\alpha), g_\alpha(\theta^\alpha)$ be two α-fractional analytic functions. Then $\big(f_\alpha(\theta^\alpha)\big)^{\otimes_\alpha m} = f_\alpha(\theta^\alpha)\otimes_\alpha \cdots \otimes_\alpha f_\alpha(\theta^\alpha)$ is called the mth power of $f_\alpha(\theta^\alpha)$. On the other hand, if $f_\alpha(\theta^\alpha)\otimes_\alpha g_\alpha(\theta^\alpha) = 1$, then $g_\alpha(\theta^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(\theta^\alpha)$, and is denoted by $\big(f_\alpha(\theta^\alpha)\big)^{\otimes_\alpha (-1)}$.

Definition 2.9 ([26])**:** The smallest positive real number T_α such that $E_\alpha(iT_\alpha) = 1$, is called the period of $E_\alpha(i\theta^\alpha)$.

Proposition 2.10 (interval recurrence formula) ([27])**:** *If* $0 < \alpha \le 1$, p, q *are real numbers,* $f_\alpha(\theta^\alpha)$ *is a* α*-fractional analytic function, then*

$$\left({}_pI_q^\alpha\right)[f_\alpha(\theta^\alpha)] = \left({}_pI_q^\alpha\right)\left[f_\alpha\left(\frac{1}{\Gamma(\alpha+1)}p^\alpha + \frac{1}{\Gamma(\alpha+1)}q^\alpha - \frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)\right]. \tag{17}$$

III. MAIN RESULTS

In this section, we use interval recurrence formula to solve two improper fractional integrals. At first, we need a lemma.

Lemma 3.1: *Suppose that* $0 < \alpha \le 1$, *then*

$$\left({}_0I^\alpha_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_\alpha\big(sin_\alpha(\theta^\alpha)\big)\right] = \left({}_0I^\alpha_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_\alpha\big(cos_\alpha(\theta^\alpha)\big)\right]. \tag{18}$$

Proof By interval recurrence formula,

$$\left({}_0I^\alpha_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_\alpha\big(sin_\alpha(\theta^\alpha)\big)\right]$$

$$= \left({}_0I^\alpha_{\left[\Gamma(\alpha+1)\cdot\frac{T_\alpha}{4}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_\alpha\left(sin_\alpha\left(\frac{T_\alpha}{4} - \frac{1}{\Gamma(\alpha+1)}\theta^\alpha\right)\right)\right]$$

$$=\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_{\alpha}}{4}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_{\alpha}\left(cos_{\alpha}(\theta^{\alpha})\right)\right].$$ Q.e.d.

Theorem 3.2: *Assume that* $0<\alpha\le 1$, *then*

$$\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_{\alpha}}{4}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_{\alpha}\left(sin_{\alpha}(\theta^{\alpha})\right)\right]=\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_{\alpha}}{4}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_{\alpha}\left(cos_{\alpha}(\theta^{\alpha})\right)\right]=-\frac{T_{\alpha}}{4}\cdot Ln_{\alpha}(2)\,. \tag{19}$$

Proof $\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_{\alpha}}{4}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_{\alpha}\left(sin_{\alpha}(\theta^{\alpha})\right)\right]$

$$=2\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_{\alpha}}{8}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_{\alpha}\left(sin_{\alpha}(2t^{\alpha})\right)\right]$$

$$=2\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_{\alpha}}{8}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_{\alpha}\left(2\cdot sin_{\alpha}(t^{\alpha})\otimes_{\alpha}cos_{\alpha}(t^{\alpha})\right)\right]$$

$$=2\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_{\alpha}}{8}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_{\alpha}(2)+Ln_{\alpha}\left(sin_{\alpha}(t^{\alpha})\right)+Ln_{\alpha}\left(cos_{\alpha}(t^{\alpha})\right)\right]$$

$$=2\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_{\alpha}}{8}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_{\alpha}(2)\right]+2\left\{\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_{\alpha}}{8}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_{\alpha}\left(sin_{\alpha}(t^{\alpha})\right)\right]+\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_{\alpha}}{8}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_{\alpha}\left(cos_{\alpha}(t^{\alpha})\right)\right]\right\}$$

$$=\frac{T_{\alpha}}{4}\cdot Ln_{\alpha}(2)+2\left\{\begin{array}{c}\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_{\alpha}}{8}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_{\alpha}\left(sin_{\alpha}(t^{\alpha})\right)\right]+\\ \left({}_{\left[\Gamma(\alpha+1)\cdot\frac{T_{\alpha}}{8}\right]^{\frac{1}{\alpha}}}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_{\alpha}}{4}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_{\alpha}\left(cos_{\alpha}\left(\frac{T_{\alpha}}{4}-\frac{1}{\Gamma(\alpha+1)}u^{\alpha}\right)\right)\right]\end{array}\right\}$$

$$=\frac{T_{\alpha}}{4}\cdot Ln_{\alpha}(2)+2\left\{\begin{array}{c}\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_{\alpha}}{8}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_{\alpha}\left(sin_{\alpha}(t^{\alpha})\right)\right]+\\ \left({}_{\left[\Gamma(\alpha+1)\cdot\frac{T_{\alpha}}{8}\right]^{\frac{1}{\alpha}}}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_{\alpha}}{4}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_{\alpha}\left(sin_{\alpha}(u^{\alpha})\right)\right]\end{array}\right\}$$

$$=\frac{T_{\alpha}}{4}\cdot Ln_{\alpha}(2)+2\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_{\alpha}}{4}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_{\alpha}\left(sin_{\alpha}(\theta^{\alpha})\right)\right]. \tag{20}$$

Therefore,

$$-\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_{\alpha}}{4}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_{\alpha}\left(sin_{\alpha}(\theta^{\alpha})\right)\right]=\frac{T_{\alpha}}{4}\cdot Ln_{\alpha}(2). \tag{21}$$

That is,

$$\left({}_{0}I^{\alpha}_{\left[\Gamma(\alpha+1)\cdot\frac{T_{\alpha}}{4}\right]^{\frac{1}{\alpha}}}\right)\left[Ln_{\alpha}\left(sin_{\alpha}(\theta^{\alpha})\right)\right]=-\frac{T_{\alpha}}{4}\cdot Ln_{\alpha}(2).$$

By Lemma 3.1, the desired result holds. Q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, we use interval recurrence formula to solve two improper fractional integrals. In fact, our result is a generalization of traditional calculus result. In the future, we will continue to use Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions to study the problems in fractional differential equations and engineering mathematics.

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